H_∞ Output Feedback Control of Constrained Systems via Moving Horizon Strategy

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Abstract This paper addresses the H_∞ output feedback control problem for discrete-time systems with actuator saturation. Initially, a constrained H_∞ output feedback control approach is presented in the framework of linear matrix inequalities (LMI) optimization. Under certain assumptions on the disturbance energy bound, closed-loop H_∞ performance is achieved. Furthermore, the moving horizon strategy is applied to an online management of the control performance so that the closed-loop system can satisfy control constraints in the case of unexpected large disturbances. A dissipation constraint is derived to achieve the moving horizon closed-loop system dissipative. Simulation results show that the constrained H_∞controller works effectively under the disturbance assumption and that the moving horizon H_∞controller can trade-off automatically between satisfying control constraints and enhancing performance.

Key words Constrained systems, H_∞ performance, output feedback, LMI optimization, moving horizon control

1 Introduction

All actuators suffer saturation because of physical limitation. Actuator saturation may result in control performance degradation, and more badly drive control systems unstable. Recently, there have been many efforts for analyzing and synthesizing control systems with actuator saturation. [1] and [2] used circle criterion and Popov′s criterion to estimate the attraction region of the constrained systems with sector-bounded actuator saturation, based on linear matrix inequalities (LMI) optimization. However, this method gave much conservative results and [3] addressed the attraction region issue of the systems with actuator saturation and disturbance, by adopting linear differential inclusion (LDI) to describe the saturated system, and present a less conservative method to estimate the attraction region. Moving horizon control, mostly named as model predictive control (MPC), can deal with time-domain constraints in an explicit and direct fashion and provides less conservative (or nonconservative) solution to constrained control[4∼6]. Combining H_∞ control with the moving horizon strategy, [7] discussed the H_∞ performance issue of linear systems with both time-domain constraints and disturbances, presented a state feedback H_∞ moving horizon control scheme. Closed-loop stability and H_∞ performance were achieved by introducing a dissipation constraint which guaranteed the moving horizon system dissipative in [8]. The above-mentioned methods are based on state feedback control law. In practice, however, not all system states are directly measured. So it is important to develop output feedback controller design approaches.

This paper considers H_∞ output feedback control problem with control constraints. In the framework of LMI optimization[9] and multiobjective control[10], an output feedback controller design approach to constrained systems, which can guarantee H_∞ performance of the close-loop system and satisfy time-domain constraints was presented. To deal with either constraint violation or conservative design, moving horizon strategy to online adjust the control performance index was used. Hence, the moving horizon closed-loop system can automatically trade-off between enhancing control performance and satisfying control constraints.

The rest of the paper is organized as follows: Section 2 describes the control problem to be considered. Section 3 presents the LMI approach for designing constrained H_∞ output feedback controllers, and closed-loop properties are discussed. Section 4 discusses the moving horizon algorithm and derives a condition to achieve the moving horizon system dissipative, as in [8]. The simulation results are presented in Section 5.

Received August 10, 2006; in revised form April 28, 2007
Supported by National Natural Science Foundation of China (60374027), Program for New Century Excellent Talents in University (2004)
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DOI: 10.1360/aas-007-1176

2 Problem statement

Consider the following discrete-time system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B_u\boldsymbol{u}(k) + B_w\boldsymbol{w}(k) \tag{1a}$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{w}(k) \tag{1b}$$
$$\boldsymbol{z}(k) = \begin{bmatrix} H\boldsymbol{x}(k) \\ \boldsymbol{u}(k) \end{bmatrix} \tag{1c}$$

where $\boldsymbol{x}(k) \in \mathbf{R}^n$ is the state vector, $\boldsymbol{y}(k) \in \mathbf{R}^m$ is the vector of measured outputs, $\boldsymbol{z}(k) \in \mathbf{R}^p$ is the vector of controlled outputs, $\boldsymbol{u}(k) \in \mathbf{R}^q$ is the vector of control inputs, $\boldsymbol{w}(k) \in \mathbf{R}^l$ is the vector of disturbance inputs. The control inputs $\boldsymbol{u}(k)$ are constrained as

$$|u_j(k)| \le u_{j,\max},\ \forall k \ge 0,\ j = 1, 2, \cdots, q \tag{2}$$

A fundamental assumption is (A, B_u) is stabilizable, (C, A) and (H, A) are detectable.

We consider the following full-order output feedback controller

$$\hat{\boldsymbol{x}}(k+1) = A_c\hat{\boldsymbol{x}}(k) + B_c\boldsymbol{y}(k) \tag{3a}$$
$$\boldsymbol{u}(k) = C_c\hat{\boldsymbol{x}}(k) \tag{3b}$$

where $\hat{\boldsymbol{x}}(k) \in \mathbf{R}^n$ is the vector of controller states. Define

$$\boldsymbol{x}_{cl} = \begin{bmatrix} \boldsymbol{x} \\ \hat{\boldsymbol{x}} \end{bmatrix} \tag{4}$$

the closed-loop system can be described as

$$\boldsymbol{x}_{cl}(k+1) = \bar{A}\boldsymbol{x}_{cl}(k) + \bar{B}\boldsymbol{w}(k) \tag{5a}$$
$$\boldsymbol{z}(k) = \bar{C}\boldsymbol{x}_{cl}(k) \tag{5b}$$

where

$$\bar{A} = \begin{bmatrix} A & B_u C_c \\ B_c C & A_c \end{bmatrix} \tag{6a}$$

$$\bar{B} = \begin{bmatrix} B_w \\ B_c D \end{bmatrix} \tag{6b}$$

$$\bar{C} = \begin{bmatrix} H & 0 \\ 0 & C_c \end{bmatrix} \tag{6c}$$

The objective of this paper is to design an internally stabilizing controller in the form of (3) such that the closed-loop system (5) achieves the following properties:

1) The control constraints in (2) are satisfied;

2) It achieves that the H_∞ performance from the disturbance $\boldsymbol{w}$ to the controlled output $\boldsymbol{z}$ is less than some $\gamma > 0$.

3 H_∞ output feedback control of constrained systems

Let us first consider the unconstrained H_∞ control problem. The closed-loop H_∞ norm from $\boldsymbol{w}$ to $\boldsymbol{z}$ is less than γ if there exist positive symmetric matrices X, Y, $\hat{P}_{11}$, $\hat{P}_{22}$ and matrices $\hat{P}_{12}$, $\hat{A}_c$, $\hat{B}_c$, $\hat{C}_c$ such that the following matrix inequalities are satisfied[11].

$$\left[\begin{array}{cc|c|cc|cc} \hat{P}_{11} & * & * & * & * & * & * \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} & * & * & * & * & * \\ \hline 0 & 0 & \gamma^2 I & * & * & * & * \\ AY + B_u\hat{C}_c & A & B_w & Y & * & * & * \\ \hat{A}_c & XA + \hat{B}_c C & XB_w + \hat{B}_c D & I & X & * & * \\ \hline HY & H & 0 & 0 & 0 & I & * \\ \hat{C}_c & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] > 0 \tag{7}$$

$$\begin{bmatrix} Y & I \\ I & X \end{bmatrix} - \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} \end{bmatrix} \geq 0 \tag{8}$$

where the symbol "$*$" represents the transpose of the related term. The associated controller matrices can be computed by

$$\begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix} = \begin{bmatrix} Q - X & XB_u \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_c - XAY & \hat{B}_c \\ \hat{C}_c & 0 \end{bmatrix} \begin{bmatrix} Y & 0 \\ CY & I \end{bmatrix}^{-1} \tag{9}$$

with $Q = Y^{-1}$. This can be briefly shown as follows. Define

$$J = \begin{bmatrix} Y & I \\ Y & 0 \end{bmatrix} \tag{10}$$

The congruence transformation with $\mathrm{diag}(J^{-1}, I, J^{-1}, I)$ and $\mathrm{diag}(J^{-1}, J^{-1})$ on (7) and (8), respectively, was performed. Note that (7) is partitioned accordingly. Then, using (6) and (9), we achieve that (7) and (8) are equivalent to each other[1]

$$\begin{bmatrix} P_1 & * & * & * \\ 0 & \gamma^2 I & * & * \\ P_2\bar{A} & P_2\bar{B} & P_2 & * \\ \bar{C} & 0 & 0 & I \end{bmatrix} > 0 \tag{11a}$$

$$P_2 \geq P_1 \tag{11b}$$

where

$$P_1 = J^{-T} \begin{bmatrix} \hat{P}_{11} & * \\ \hat{P}_{12}^{\mathrm{T}} & \hat{P}_{22} \end{bmatrix} J^{-1} \tag{12a}$$

$$P_2 = \begin{bmatrix} X & Q - X \\ Q - X & X - Q \end{bmatrix} \tag{12b}$$

By the use of Schur complement, (11a) is equivalent to

$$\begin{bmatrix} P_1 & 0 \\ 0 & \gamma^2 I \end{bmatrix} - \begin{bmatrix} \bar{A}^{\mathrm{T}} P_2 \bar{A} + \bar{C}^{\mathrm{T}}\bar{C} & \bar{A}^{\mathrm{T}} P_2 \bar{B} \\ \bar{B}^{\mathrm{T}} P_2 \bar{A} & \bar{B}^{\mathrm{T}} P_2 \bar{B} \end{bmatrix} > 0 \tag{13}$$

which implies

$$\begin{bmatrix} \boldsymbol{x}_{cl}(i) \\ \boldsymbol{w}(i) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{A}^{\mathrm{T}} P_2 \bar{A} + \bar{C}^{\mathrm{T}}\bar{C} - P_1 & \bar{A}^{\mathrm{T}} P_2 \bar{B} \\ \bar{B}^{\mathrm{T}} P_2 \bar{A} & \bar{B}^{\mathrm{T}} P_2 \bar{B} - \gamma^2 I \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{x}_{cl}(i) \\ \boldsymbol{w}(i) \end{bmatrix} \leq 0 \tag{14}$$

for any $\boldsymbol{x}_{cl}(i)$ and $\boldsymbol{w}(i)$. By using (5), we obtain

$$\boldsymbol{x}_{cl}^{\mathrm{T}}(i+1) P_2 \boldsymbol{x}_{cl}(i+1) + \|\boldsymbol{z}(i)\|^2 \leq \gamma^2 \|\boldsymbol{w}(i)\|^2 + \boldsymbol{x}_{cl}^{\mathrm{T}}(i) P_1 \boldsymbol{x}_{cl}(i) \tag{15}$$

Summing up (15) from $i = 0$ to $i = k$ leads to

$$\boldsymbol{x}_{cl}^{\mathrm{T}}(k+1) P_2 \boldsymbol{x}_{cl}(k+1) + \sum_{i=0}^{k} \|\boldsymbol{z}(k)\|^2 \leq \gamma^2 \sum_{i=0}^{k} \|\boldsymbol{w}(k)\|^2 + \boldsymbol{x}_{cl}^{\mathrm{T}}(0) P_1 \boldsymbol{x}_{cl}(0) - \sum_{i=1}^{k} \boldsymbol{x}_{cl}^{\mathrm{T}}(k)(P_2 - P_1)\boldsymbol{x}_{cl}(k) \tag{16}$$

Because of $P_2 \geq P_1$, we can conclude that the closed-loop system obeys

$$\boldsymbol{x}_{cl}^{\mathrm{T}}(k+1) P_2 \boldsymbol{x}_{cl}(k+1) + \sum_{i=0}^{k} \|\boldsymbol{z}(k)\|^2 \leq \gamma^2 \sum_{i=0}^{k} \|\boldsymbol{w}(k)\|^2 + \boldsymbol{x}_{cl}^{\mathrm{T}}(0) P_1 \boldsymbol{x}_{cl}(0) \tag{17}$$

and hence achieve the H_∞ norm from $\boldsymbol{w}$ to $\boldsymbol{z}$ is less than γ. Moreover, the feasibility of (13) leads to $\bar{A}^{\mathrm{T}} P_2 \bar{A} - P_1 < 0$ and furthermore $\bar{A}^{\mathrm{T}} P_2 \bar{A} - P_2 < 0$ due to $P_2 \geq P_1$. This implies that (5) is quadratically stable because of $P_2 > 0$.

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[1] It follows from (9) that $\hat{C}_c = C_c Y$, $\hat{B}_c = (Q - X)B_c$ and $\hat{A}_c Q = (XA + \hat{B}_c C) + (Q - X)A_c + XB_u C_c$. Then, we can show for example that the block (3,1) in (7) equals to the block (3,1) in (11a) by $J^{-\mathrm{T}} \begin{bmatrix} AY + B_u\hat{C}_c & A \\ \hat{A}_c & XA + \hat{B}_c C \end{bmatrix} J^{-1} = \begin{bmatrix} XA + \hat{B}_c C & \hat{A}_c Q - (XA + \hat{B}_c C) \\ Q^{\mathrm{T}} A - (XA + \hat{B}_c C) & Q^{\mathrm{T}} B_u \hat{C}_c Q - \hat{A}_c Q + XA + \hat{B}_c C \end{bmatrix}$ and

$$P_2\bar{A} = \begin{bmatrix} XA + (Q - X)B_c C & XB_u C_c + (Q - X)A_c \\ (Q^{\mathrm{T}} - X)A - (Q - X)B_c C & (Q^{\mathrm{T}} - X)B_u C_c + (X - Q)A_c \end{bmatrix}.$$

Now, we discuss the constrained H_∞ output feedback control problem. With $\boldsymbol{u} = C_c\hat{\boldsymbol{x}}$, the satisfaction of control constraints requires for $j = 1, \ldots, q$ that

$$\max_{k\geq 0} |u_j(k)|^2 = \max_{k\geq 0} |\boldsymbol{e}_j^{\mathrm{T}} \hat{C}_c Q\hat{\boldsymbol{x}}(k)|^2 \leq u_{j,\max}^2 \tag{18}$$

where $\boldsymbol{e}_j$ is the standard basis vectors in $\mathbf{R}^q$. This implies that constraints on $\boldsymbol{u}$ can be enforced by constraining $\hat{\boldsymbol{x}}$. Considering (12b), we infer

$$\begin{aligned}
&\boldsymbol{x}_{cl}^{\mathrm{T}} P_2 \boldsymbol{x}_{cl} = \\
&\boldsymbol{x}^{\mathrm{T}} X\boldsymbol{x} + \hat{\boldsymbol{x}}^{\mathrm{T}}(Q-X)\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}(Q-X)\hat{\boldsymbol{x}} + \boldsymbol{x}^{\mathrm{T}}(X-Q)\boldsymbol{x} = \\
&\hat{\boldsymbol{x}}^{\mathrm{T}} R\hat{\boldsymbol{x}} + \boldsymbol{x}^{\mathrm{T}} X\boldsymbol{x} - \hat{\boldsymbol{x}}^{\mathrm{T}}(I - QX^{-1})X\boldsymbol{x} - \\
&\boldsymbol{x}^{\mathrm{T}} X(X^{-1}Q - I)\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}^{\mathrm{T}}(X-Q)(I - X^{-1}Q)\hat{\boldsymbol{x}} = \\
&\hat{\boldsymbol{x}}^{\mathrm{T}} R\hat{\boldsymbol{x}} + d(\boldsymbol{x}, \hat{\boldsymbol{x}})
\end{aligned} \tag{19}$$

where $R = Q - QX^{-1}Q$ and $d(\boldsymbol{x}, \hat{\boldsymbol{x}}) = [\boldsymbol{x} - (I - X^{-1}Q)\hat{\boldsymbol{x}}]^{\mathrm{T}} X[\boldsymbol{x} - (I - X^{-1}Q)\hat{\boldsymbol{x}}]$. Thus, it follows from (17) that

$$\begin{aligned}
&\hat{\boldsymbol{x}}^{\mathrm{T}}(k+1)R\hat{\boldsymbol{x}}(k+1) + \sum_{i=0}^{k} \|\boldsymbol{z}(k)\|^2 \leq \gamma^2 \sum_{i=0}^{k} \|\boldsymbol{w}(k)\|^2 + \\
&\qquad \boldsymbol{x}_{cl}^{\mathrm{T}}(0)P_1\boldsymbol{x}_{cl}(0) - d(\boldsymbol{x}(k+1), \hat{\boldsymbol{x}}(k+1))
\end{aligned} \tag{20}$$

Because $d(\boldsymbol{x}, \hat{\boldsymbol{x}}) \geq 0$, (20) implies that if the initial state $\boldsymbol{x}_{cl}(0)$ belongs to an ellipsoid Ω_1 defined by

$$\Omega_1(P_1, r, w_{\max}) = \{\boldsymbol{x}_{cl} \in \mathbf{R}^{2n} : \boldsymbol{x}_{cl}^{\mathrm{T}} P_1 \boldsymbol{x}_{cl} + \gamma^2 w_{\max} \leq r\} \tag{21}$$

then, the controller state trajectory stays in an another ellipsoid Ω_2 defined by

$$\Omega_2(R, r) = \{\hat{\boldsymbol{x}} \in \mathbf{R}^n : \hat{\boldsymbol{x}}^{\mathrm{T}} R\hat{\boldsymbol{x}} \leq r\} \tag{22}$$

i.e., $\hat{\boldsymbol{x}}(k) \in \Omega_2(R, r)$ for all $k \geq 0$. Hence, we infer

$$\begin{aligned}
&\max_{k\geq 0} |\boldsymbol{e}_j^{\mathrm{T}} \hat{C}_c Q\hat{\boldsymbol{x}}(k)|^2 \leq \max_{\hat{\boldsymbol{x}}\in\Omega_2(R,r)} |\boldsymbol{e}_j^{\mathrm{T}} \hat{C}_c Q\hat{\boldsymbol{x}}|^2 \leq \\
&r\|\boldsymbol{e}_j^{\mathrm{T}} \hat{C}_c QR^{-\frac{1}{2}}\|_2^2 = r\boldsymbol{e}_j^{\mathrm{T}} \hat{C}_c QR^{-1}Q\hat{C}_c^{\mathrm{T}} \boldsymbol{e}_j = \\
&r\boldsymbol{e}_j^{\mathrm{T}} \hat{C}_c (Y - X^{-1})^{-1} \hat{C}_c^{\mathrm{T}} \boldsymbol{e}_j
\end{aligned} \tag{23}$$

and conclude that (18) can be satisfied by enforcing

$$\begin{bmatrix} \frac{u_{j,\max}^2}{r} & \boldsymbol{e}_j^{\mathrm{T}} \hat{C}_c & * \\ * & Y & * \\ 0 & I & X \end{bmatrix} \geq 0, \; j = 1, 2, \ldots, q \tag{24}$$

which are LMIs for fixed r. This implies that if $\boldsymbol{x}_{cl}(0) \in \Omega_1(P_1, r, w_{\max})$, then the feasibility of (24) guarantees control constraints in (2). Finally, we can solve the following LMI optimization problem

$$\min_{\gamma^2, Y, X, \hat{P}_{11}, \hat{P}_{12}, \hat{P}_{22}, \hat{A}_c, \hat{B}_c, \hat{C}_c} \gamma^2 \text{ s.t. (7), (8), and (24)} \tag{25}$$

and compute the matrices of the output feedback controller by (9). The following result for the closed-loop system can then be summarized from the above discussion.

Theorem 1. For given $r = r_o$, suppose that

1) The optimization problem (25) admits an (almost) optimal solution, denoted with the subscript "o";

2) The disturbance energy is bounded as follows.

$$w_{\max,o} = \frac{r_o - \boldsymbol{x}_{cl}^{\mathrm{T}}(0)P_{1o}\boldsymbol{x}_{cl}(0)}{\gamma_o^2} \tag{26}$$

Then, the output feedback controller constructed by (3) with (9) guarantees the closed-loop system has the following properties:

a) Control constraints in (2) are satisfied;

b) It is quadratically stable;

c) The H_∞ performance from $\boldsymbol{w}$ to controlled output $\boldsymbol{z}$ is less than γ_o;

d) The output energy is bounded by r_0.

Proof. Properties a) ∼ c) are indicated from the discussion from (10) to (24) and property d) follows from (20) because of $R > 0$. □

Theorem 1 shows that the output feedback controller constructed by solving the LMI optimization problem (25) can meet the design requirements listed in Section 2. Here, the choice of the controller parameter r is crucial to achieve the desired performance and some points about how to choose r in the following is listed.

Remark 1. The smaller the value of r, the smaller the output energy, which implies better performance. From the structure of (24), we also conclude that the larger the value of r, the smaller the set of all $(X, Y, \hat{C}_c)$ satisfying (24) and hence the larger the optimal value γ. This implies worse performance. Moreover, too large values of r might lead to infeasibility of (24) and hence infeasibility of (25). However, we cannot conclude for r that the smaller the better, because small values of r implies small volumes of Ω_2 (for fixed R)[9] and hence may result in constraint violation.

Remark 2. For simplicity, we usually assume $\boldsymbol{x}_{cl}(0) = 0$. Hence, r should be chosen to satisfy $\gamma^2 w_{\max} \leq r$.

4 Moving horizon algorithm of H_∞ output feedback control

As clarified in Remark 1, the choice of the parameter r reflects an inherent trade-off between satisfying time-domain constraints and achieving high performance. If having to be prepared for unforeseen large disturbances one has to choose a large value of r. This leads to large γ and low performance. On the other hand, enforcing high performance levels (small γ) requires to reduce r, which might result in control constraint violation in case that the system is affected by unexpectedly large disturbances. This motivates the use of the moving horizon strategy and the online solution of the optimization problem at each time. As a benefit, we achieve an automatic trade-off between the satisfaction of constraints and the level of performance by online minimization of the performance index, and it might be no longer necessary to manually adjust the parameter r.

Using the moving horizon strategy, we need to solve the above constrained H_∞ control problem online at each time k. From Section 3, we know that if $\hat{\boldsymbol{x}}(k) \in \Omega_2(R, r)$ and there are $(Y, X, \hat{C}_c)$ satisfying (24), then we can obtain

$$|u_j(k)| = |\boldsymbol{e}_j^{\mathrm{T}} C_c \hat{\boldsymbol{x}}(k)| \leq u_{j,\max} \tag{27}$$

which implies that the controller satisfies control constraints at time k. Because $\hat{\boldsymbol{x}}^{\mathrm{T}}(k)R\hat{\boldsymbol{x}}(k) \leq r$ is not an LMI on variables $(X, Y, \hat{C}_c)$, we replace it conservatively by

$$\begin{bmatrix} r & \hat{\boldsymbol{x}}^{\mathrm{T}}(k) \\ \hat{\boldsymbol{x}}(k) & Y \end{bmatrix} \geq 0 \tag{28}$$

According to moving horizon strategy, we would then solve the following LMI optimization problem at each time k:

$$\min_{\gamma^2,Y,X,\hat{P}_{11},\hat{P}_{12},\hat{P}_{22},\hat{A}_c,\hat{B}_c,\hat{C}_c} \gamma^2 \text{ s.t. (7), (8), (24), and (28)} \tag{29}$$

If (29) admits an (almost) solution and we use the subscript "k" to denote the solution at time k, then, the feasibility of (7) implies

$$\begin{aligned}&\boldsymbol{x}_{cl}^{\mathrm{T}}(k+1)P_{2,k}\boldsymbol{x}_{cl}(k+1)+\|\boldsymbol{z}(k)\|^2\leq\\&\gamma_k^2\|\boldsymbol{w}(k)\|^2+\boldsymbol{x}_{cl}^{\mathrm{T}}(k)P_{1,k}\boldsymbol{x}_{cl}(k)\end{aligned} \tag{30}$$

for the closed-loop system (5). By summing up (30) from $k=0$ to $k=l$, we obtain

$$\begin{aligned}&\sum_{k=0}^{l}\|\boldsymbol{z}(k)\|^2-\sum_{k=0}^{l}\gamma_k^2\|\boldsymbol{w}(k)\|^2\leq\boldsymbol{x}_{cl}^{\mathrm{T}}(0)P_{1,0}\boldsymbol{x}_{cl}(0)-\\&\boldsymbol{x}_{cl}^{\mathrm{T}}(l+1)P_{2,l}\boldsymbol{x}_{cl}(l+1)-\\&\sum_{k=1}^{l}\boldsymbol{x}_{cl}^{\mathrm{T}}(k)(P_{2,k-1}-P_{1,k})\boldsymbol{x}_{cl}(k)\end{aligned} \tag{31}$$

If we furthermore enforce

$$P_{2,k-1}-P_{1,k}\geq 0 \tag{32}$$

then, it follows from (31) that system (5) obeys

$$\begin{aligned}&\sum_{k=0}^{l}\|\boldsymbol{z}(k)\|^2-\gamma_{mh}^2\sum_{k=0}^{l}\|\boldsymbol{w}(k)\|^2\leq\boldsymbol{x}_{cl}^{\mathrm{T}}(0)P_{1,0}\boldsymbol{x}_{cl}(0)-\\&\boldsymbol{x}_{cl}^{\mathrm{T}}(l+1)P_{2,l}\boldsymbol{x}_{cl}(l+1)\end{aligned} \tag{33}$$

for all $l\geq 0$ and furthermore,

$$\sum_{k=0}^{l}\|\boldsymbol{z}(k)\|^2-\gamma_{mh}^2\sum_{k=0}^{l}\|\boldsymbol{w}(k)\|^2\leq\boldsymbol{x}_{cl}^{\mathrm{T}}(0)P_{1,0}\boldsymbol{x}_{cl}(0) \tag{34}$$

because of $P_2>0$, where $\gamma_{mh}=\max(\gamma_0,\cdots,\gamma_l)$. Thus, we show with the help of (32) that the output feedback moving horizon system is dissipative. As in [8], we name (32) as dissipation constraint and introduce it into the optimization problem, which will be solved at each time k. We stress that $P_{1,k}$ is to be determined at time k and $P_{2,k-1}$ is known from the previous time. Hence, (32) is transformed in the form of

$$\begin{bmatrix}\hat{P}_{11}-YQ_{k-1}Y & \hat{P}_{12}-YQ_{k-1}\\ \hat{P}_{12}^{\mathrm{T}}-Q_{k-1}Y & \hat{P}_{22}-X_{k-1}\end{bmatrix}\leq 0 \tag{35}$$

which is, however, not an LMI because of the quadratic form of $YQ_{k-1}Y$. Consulting $Q_{k-1}Y_{k-1}=I$, we enforce (35) by the following LMIs

$$\begin{bmatrix}\hat{P}_{11}-\lambda Y & \hat{P}_{12}-YQ_{k-1}\\ \hat{P}_{12}^{\mathrm{T}}-Q_{k-1}Y & \hat{P}_{22}-X_{k-1}\end{bmatrix}\leq 0,\ Y\geq\lambda Y_{k-1} \tag{36}$$

where $\lambda\in(0,1]$. Then, the optimization problem is reformulated as

$$\begin{aligned}&\min_{\gamma^2,Y,X,\hat{P}_{11},\hat{P}_{12},\hat{P}_{22},\hat{A}_c,\hat{B}_c,\hat{C}_c} \gamma^2\\&\text{s.t. (7), (8), (24), (28), and (36)}\end{aligned} \tag{37}$$

and the following moving horizon output feedback controller algorithm is suggested.

Algorithm.

Step 1. Initialization. Given r and λ.

Step 2. At time $k=0$, solve the LMI optimization problem (29) to obtain (γ_0,Y_0,X_0), $(\hat{P}_{11,0},\hat{P}_{12,0},\hat{P}_{22,0})$, and $(\hat{A}_{c,0},\hat{B}_{c,0},\hat{C}_{c,0})$. Go to Step 4.

Step 3. At time $k>0$, get $\hat{\boldsymbol{x}}(k)$ and solve the optimization problem (37) to obtain (γ_k,Y_k,X_k), $(\hat{P}_{11,k}$, $\hat{P}_{12,k}$, $\hat{P}_{22,k})$, and $(\hat{A}_{c,k},\hat{B}_{c,k},\hat{C}_{c,k})$.

Step 4. Compute $(A_{c,k},B_{c,k},C_{c,k})$ by (9) and furthermore the closed-loop control by (3) and inject into the system. Replace k by $k+1$ and continue with Step 3.

We stress that the optimization problem (37) is updated with the controller state and hence implicitly with the measurement. The online minimization of the H_∞ index makes it possible to automatically relax the performance requirement if necessary for satisfying control constraints and to recover performance when the system is far from the bounds. This might be done better if the parameter r can be adjusted online. However, the automatic management between performance and control constraints is achieved by minimizing the performance index online.

By using (34) and detectability of (H,A), we then conclude that the moving horizon H_∞ output feedback controller algorithm guarantees that the closed-loop system achieves the following properties:

1) It is asymptotically stable;
2) The control constraints are satisfied;
3) The H_∞ performance from the disturbance $\boldsymbol{w}$ to the controlled output $\boldsymbol{z}$ is less than γ_{mh}.

Remark 3. For achieving the dissipation property we introduce $P_{2,k-1}\geq P_{1,k}$. This condition is less restrictive than $P_{2,k-1}\geq P_{2,k}$ and $P_{1,k-1}\geq P_{1,k}$ as imposed in [12].

Remark 4. The introduction of constraint (28) enforces the actual controller state $\hat{\boldsymbol{x}}(k)$ to be in $\Omega_2(R,r)$ and hence enforces to satisfy control constraints. We notice that small values of r might lead to its infeasibility and hence infeasibility of (37). Hence, in Step 3 of the moving horizon algorithm, one may enlarge r to recover the feasibility of the optimization problem. However, it is limited by (24). For an effective method to recover feasibility, please refer to [13].

5 Simulation results

As a numerical example, we consider the following system

$$\boldsymbol{x}(k+1)=\begin{bmatrix}-0.8 & 0.3\\ 0.1 & 1.2\end{bmatrix}\boldsymbol{x}(k)+\begin{bmatrix}0.3\\0.1\end{bmatrix}\boldsymbol{w}(k)+\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\boldsymbol{u}(k)$$

$$\boldsymbol{y}(k)=\begin{bmatrix}-1 & 1\\ 0.5 & 1\end{bmatrix}\boldsymbol{x}(k)+\begin{bmatrix}0.2\\0.1\end{bmatrix}\boldsymbol{w}(k),\ \boldsymbol{z}(k)=\begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{u}(k)\end{bmatrix}$$

where the control input satisfies

$$|u_j(k)|\leq 0.1,\ j=1,2$$

The control objective is to achieve an H_∞ performance of $\gamma\leq 1.74$ for the disturbances whose energy is bounded by 10, *i.e.*, $\sum_{k=0}^{\infty}\|\boldsymbol{w}(k)\|^2\leq 10$, while satisfying the above control constraints. Hence, we choose $r=20$ and $\lambda=0.9$, and solved the LMI optimization problem (25) to obtain a constrained H_∞ controller with $\gamma_o=1.39$. For this fixed controller, we compute from (26) that $w_{\max,o}=10.29$ if

$\boldsymbol{x}_{cl}(0) = 0$, which implies the satisfaction of the design requirements. Simulation results are shown in Fig. 1, where the energy of each disturbance impulse is about 10.0 less than $w_{\max,o}$. Note that we mix some noises in disturbances.

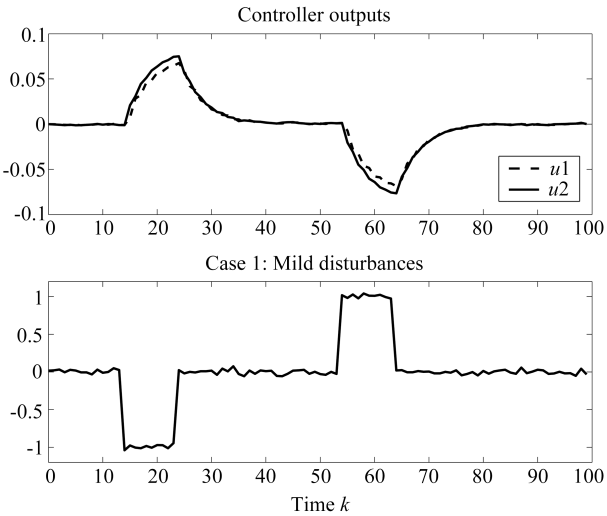

Fig. 1 Results of constrained H_∞ output feedback controller (Case 1: Mild disturbances)

Fig. 2 presents the results, where the second disturbance impulse is unexpectedly large, *i.e.*, the energy is 28.43 which exceeds the designed bound. It is clear that the fixed controller violates the control bounds, which are indicated by the dotted lines. Hence, the moving horizon H_∞ output feedback algorithm proposed in Section 4 is applied. The results are plotted in Fig. 3, where the same parameters are chosen, *i.e.*, $r = 20$ and $\lambda = 0.9$. It is clear that the control constraints are satisfied. From the curve of γ, plotted in the bottom of Fig. 3, we see that during the effect of the second disturbance impulse, the moving horizon controller relaxes automatically the performance index γ to satisfy control constraints. After the large disturbance disappears, the performance index recovers.

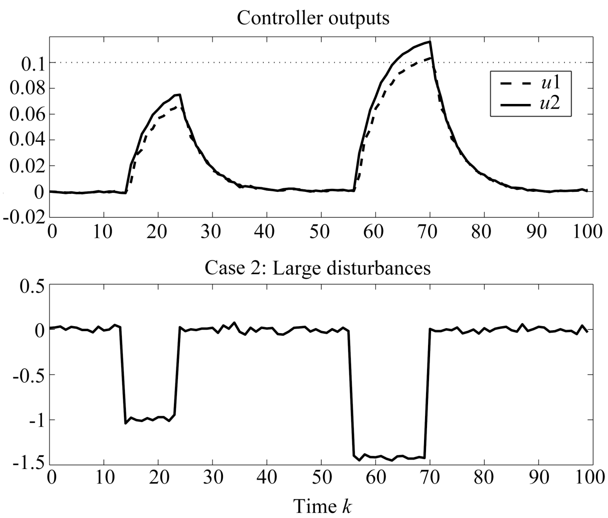

Fig. 2 Results of constrained H_∞ output feedback controller (Case 2: Unexpectedly large disturbances)

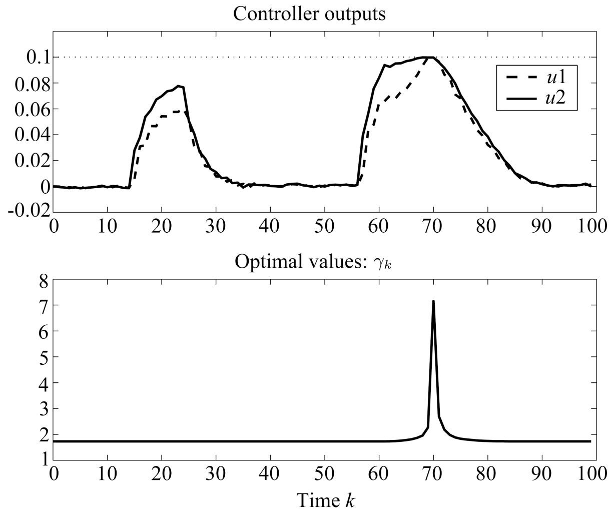

Fig. 3 Results of moving horizon H_∞ output feedback controller

6 Conclusion

In the framework of LMI optimization, this paper has proposed an output feedback H_∞ control approach to linear systems with actuator saturation. To deal with either constraint violation or conservative design, moving horizon strategy is applied and the LMI optimization problem is solved online at each sampling time. A dissipation constraint is introduced to achieve the moving horizon closed-loop system dissipative, and hence H_∞ performance. By minimizing the H_∞ level online, the moving horizon system is able to relax the performance index automatically such that the control constraints can be satisfied. This happens when unforeseen large disturbances drive the system approaching bounds. The performance recovers after large disturbances vanish.

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